The Coupling of the Parker Instability and the Balbus-Hawley Instability: An Important Structural Agency in the Stellar and Planetary Formation

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3rd International Conference on
LABORATORY ASTROPHYSICS
WITH INTENSE LASERS

Rice University
Houston, Texas
March 30 -April 1, 2000

Issues and Outline

- Collisionless, radial transport of angular momentum is observed and required for both

  (i) tokamak plasmas

  (ii) astrophysical accretion disks

- Neutral fluid experiments show onset of transport by axisymmetric Taylor vortices for $\Omega(r) = \Omega_1(r_1/r)^q$ with $q > 2$ (Koschmeider and Swinney).

- MHD theory predicts the frozen-in magnetic field changes critical gradient to $q > 0$ in weak-B-field limit (Chandrasekhar, (p. 385-387)-Velikhov.

- Turbulent simulations (Balbus-Hawley)

- Here we give turbulence theory and present 3D MHD simulations on a grid and in pseudo-spectral + radial grid code.

Turbulence and Structures in Protoplanetary Disk

Key issue defined by Kuiper, Cameron, and Boss

1. Can instabilities in the protoplanetary disk produce condensation of gas into clumps $\lesssim 10^6$ yrs.?
2. Can the Parker instability subsequently, in a symbiotic way, eject the remaining gas to create Extrasolar systems?

$\therefore$ Need better understanding through theory and simulations of disk turbulence.

Boussinesq MHD Equations

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=\mathbf{j}\times\mathbf{B}-\nabla p-\rho\nabla\Phi_G(r) \tag{1}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B}). \tag{2}$$

Rotating Equilibrium

$$\widehat{r} \qquad -\rho r\Omega^2=-\rho G(r)\cong-\rho\,\frac{MG}{r^2}$$

$$\widehat{z} \qquad 0=-\frac{dp}{dz}-\rho g_z$$

$$\Omega(r)=\Omega_1\left(\frac{r_1}{r}\right)^{3/2} \tag{3}$$

$$g_z=\left(\frac{MG}{r^3}\right)z$$

$$p=p_0(r)\left(1-\frac{z^2}{L_p^2}\right)$$

Turbulent State

$$
\begin{aligned}
\mathbf{v} &= r\Omega(r)\widehat{\mathbf{e}}_\theta + \mathbf{v} + \mathbf{u} \\
\mathbf{B} &= B_z\widehat{\mathbf{e}}_z + 0 + \delta\mathbf{B}_\perp \\
P &= P(z) + \delta p + 0
\end{aligned}
\tag{4}
$$

$$\text{Field} = \text{Eq} \ + \ \text{large scale} + \text{small scale}$$

$$r_1, L_p \qquad q, \nu + i\Gamma \qquad k, \omega + i\gamma$$

Scale Separation $k \gg q$

Equations (4) → (1) and (2) with scale separation

$$\mathbf{j} \times \mathbf{B} - \nabla p = \frac{1}{4\pi}(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B} - \nabla\left(p + \frac{B^2}{8\pi}\right) \tag{5}$$

$$\mathbf{v} \cdot \nabla \mathbf{v} = \nabla\, \frac{v^2}{2} - \mathbf{v} \times (\nabla \times \mathbf{v}) \tag{6}$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \qquad \mathbf{j} = \frac{\boldsymbol{\nabla} \times \mathbf{B}}{4\pi} \tag{7}$$

$$\frac{\partial \mathbf{v}_L}{\partial t} - \mathbf{v}_L \times \boldsymbol{\omega}_L - \langle \mathbf{u}_s \times \boldsymbol{\omega}_s \rangle = \frac{\langle \mathbf{B}_s \cdot \nabla \mathbf{B}_s \rangle}{4\pi\rho}$$

$$- \frac{\delta\rho_L}{\rho}\, g\widehat{\mathbf{z}} \tag{8}$$

$$\frac{\partial \mathbf{u}_s}{\partial t} - \mathbf{u}_s \times \boldsymbol{\omega}_s - \mathbf{v}_L \times \boldsymbol{\omega}_s - \mathbf{v}_s \times \boldsymbol{\omega}_L$$

$$= \frac{\mathbf{B}_L \cdot \nabla \mathbf{B}_s}{4\pi\rho} - \nabla\left(\frac{u_s^2}{2} + \frac{B^2}{8\pi\rho} + \frac{p}{\rho}\right) \tag{9}$$

$$\frac{\partial \mathbf{B}_L}{\partial t} = \nabla \times [\mathbf{v}_L \times \mathbf{B}_L + \langle \mathbf{u}_s \times \mathbf{B}_s \rangle] \tag{10}$$

$$\frac{\partial \mathbf{B}_s}{\partial t} = \nabla \times \left(\mathbf{v}_L \times \mathbf{B}_s + \mathbf{u}_s \times \mathbf{B}_L + \mathbf{u}_s \times \mathbf{B}_s \right). \quad (11)$$

Now reduce (8) and (9) with

$$\boldsymbol{\nabla} \cdot \mathbf{v}_L = 0 \qquad \boldsymbol{\nabla} \cdot \mathbf{u}_s = 0 \qquad (12)$$

which take the rotational parts of Eqs. (8) and (9) with

$$\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\boldsymbol{\omega}. \qquad (13)$$

Small Scale Turbulence Driven by the Keplerian Shear Flow

The small scale fluctuations satisfy

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v}_L)\,\delta\mathbf{B}_s = (\mathbf{B}_L\cdot\nabla)\mathbf{u}_s + \mathbf{B}_s\cdot\nabla\mathbf{v}_L(r)$$

giving, $\overline{\omega} = \omega - \mathbf{k}\cdot\mathbf{v}_L$,

$$-i\overline{\omega}\delta B_r = ikBu_r \tag{1}$$

$$-i\overline{\omega}\delta B_\theta = ikBu_\theta + r\,\frac{d\Omega}{dr}\,\delta B_r. \tag{2}$$

The small scale velocity fluctuations satisfy, to the lowest order in $k_z\Delta r \gg 1$,

$$-i\overline{\omega}u_r - 2\Omega u_\theta = \frac{ikB}{4\pi\rho}\,\delta B_r \tag{3}$$

$$-i\overline{\omega}u_\theta + \left(2\Omega + r\,\frac{d\Omega}{dr}\right)u_r = \frac{ikB}{4\pi\rho}\,\delta B_\theta. \tag{4}$$

In Eqs. (3) and (4) we neglect the contributions $\nabla\delta p/\rho$ which are determined from $\nabla\cdot\mathbf{u}_s$ and give the radial eigenvalue equation.

Local Shearing Dispersion Relation

Eliminating the magnetic fluctuations in Eqs. (1)–(4) yields the local relations

$$\begin{bmatrix} -i\overline{\omega} + i\,\dfrac{\omega_A^2}{\overline{\omega}} & -2\Omega \\ 2\Omega + r\,\dfrac{d\Omega}{dr} - \dfrac{\omega_A^2}{\overline{\omega}^2}\,r\,\dfrac{d\Omega}{dr} & -i\overline{\omega} + \dfrac{i\omega_A^2}{\overline{\omega}} \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = 0$$

which requires for $u_s \neq 0$ that

$$D(\omega, r) = \left(\overline{\omega} - \frac{\omega_A^2}{\overline{\omega}}\right)^2 - 2\Omega\left[2\Omega + r\Omega' - \frac{\omega_A^2}{\overline{\omega}^2}\,r\Omega'\right] = 0 \tag{5}$$

$$= \overline{\omega}^4 - \overline{\omega}^2\left[2\Omega(2\Omega + r\Omega') + 2\omega_A^2\right] + \omega_A^2\left[\omega_A^2 + 2r\Omega\,\frac{d\Omega}{dr}\right] = 0. \tag{6}$$

This dispersion relation is the same as given by Balbus–Hawley in Eq. (111) and in Chandrasekhar pp. 385–387.

Local Shearing Dispersion Relation

For $\omega_A^2/\Omega^2 \to 0$ the dispersion relation has the Couette flow modes $\omega^2 = 0$ and $\omega^2 = 2\Omega\, d(r^2\Omega)/dr$ which is stable for monotonically increasing values of $r^2\Omega(r)$. Now for $\omega_A^2/\Omega^2 \neq 0$ the zero frequency modes are destabilized for any $d\Omega/dr < 0$ for sufficiently small $\omega_A^2 \propto k_z^2 v_A^2$. The practical limit of small ω_A is given by B_z and $k_z \geq \pi/H$ where H is the height of the accretion disk. Thus, we have for

$$\boxed{-r\,\frac{d\Omega^2}{dr} > \omega_A^2 \geq \frac{B_z^2}{4\pi\rho}\left(\frac{\pi}{H}\right)^2} \tag{7}$$

Purely growing modes $\overline{\omega} = \pm i\gamma$ in the frame of plasma with

$$\gamma^2 \cong \frac{-\omega_A^2(\omega_A^2 + r\,\frac{d\Omega^2}{dr})}{2\omega_A^2 + 4\Omega^2 + r\,\frac{d\Omega^2}{dr}} \simeq \frac{\omega_A^2(3\Omega^2 - \omega_A^2)}{2\omega_A^2 + \Omega^2}. \tag{8}$$

Maximum Growth Rate

Computing $d\gamma/dk_z$ we find the maximum growth rate at $\omega_A^4 + (4-2q)\omega_A^2 - q(4-2q) = 0$. For $q = 3/2$ this gives

$$\frac{k_z v_A}{\Omega} = \left(\frac{7^{1/2}-1}{2}\right)^{1/2} = 0.91$$

with

$$\gamma_{\max} = \frac{\omega_A(2q-\omega_A^2)^{1/2}}{(4-2q+2\omega_A^2)^{1/2}} = 0.823\Omega$$

for Keplerian flow. For the full dispersion relation Fig. 1 shows the exact growth $\gamma(k_z)$ compared to the values in Eq. (9) and (9) obtained the weak coupling limit of the low frequency ω_A^2 compared with the high frequency Ω^2.

The pressure fluctuation δp is 90° out of phase with u_z since

$$-i\varpi\, u_z = -ik_z\,\frac{\delta p}{\rho} + \frac{ik_zB}{4\pi\rho}\,\delta B_z \tag{9}$$

and

$$\frac{(\overline{\omega}^2 - \omega_A^2)}{\overline{\omega}} \, \frac{1}{r} \, \frac{\partial}{\partial r} \, (r u_r) = -i k_z^2 \, \frac{\delta p}{\rho}. \tag{10}$$

Using Eq. (9) to calculate the correction to the right–hand side of Eq. (10) from the radial gradient of the pressure fluctuation, we get

$$\frac{\partial}{\partial r} \left[\frac{\overline{\omega}^2 - \omega_A^2}{\overline{\omega}} \, \frac{1}{r} \, \frac{\partial}{\partial r} \, (r u_r) \right] + k_z^2$$

$$\times \left[-\frac{\overline{\omega}^2 - \omega_A^2}{\overline{\omega}} + \frac{2\Omega(2\Omega + r\Omega' - \omega_A^2/\overline{\omega}^2 \, r\Omega')}{\overline{\omega} - \omega_A^2/\overline{\omega}} \right] u_r = 0. \tag{11}$$

Now for $m = 0$ we have $\overline{\omega}^2 = \text{const}$ and taking ω_A^2 constant, Eq. (11) reduces to

$$\left\{ K D D^* + k_z^2 \left[-K + \frac{4\overline{\omega}^2 \Omega^2}{\overline{\omega}^2 - \omega_A^2} + 2r\Omega\Omega' \right] \right\} u_r = 0 \tag{12}$$

where $K = \overline{\omega}^2 - \omega_A^2$ are $D = \partial_r$ and $D_* = \partial_r + 1/r$. This is Eq. (52) on p. 386 of Chandrasekhar where he uses $\Phi = 4\Omega^2 + 2\Omega \, r\Omega'$ to rewrite (12) as

$$K \left(D \, D^* - k_z^2 \right) = k_z^2 \left[\Phi + \frac{4\Omega^2 \omega_A^2}{K} \right] \tag{13}$$

operating on ξ_r instead of u_r.

Following Chandrasekhar, we define

$$I_1 = \int_{r_1}^{r_2} \left[(D_* \xi_r)^2 + k_z^2 \xi_r^2 \right] r dr$$

$$I_2 = \int_{r_1}^{r_2} \Phi \xi_r^2 r dr$$

$$I_3 = \int_{r_1}^{r_2} \Omega^2 \xi_r^2 r dr$$

and get the stability condition that replaces Eq. (7) as

$$\boxed{-r \, \frac{B^2}{4\pi\rho} \, I_1 > -\int_{r_1}^{r_2} \frac{d\Omega^2}{dr} \, r^2 \xi^2 r dr.} \qquad (14)$$

Thus, we see that $k_z^2 \rightarrow k_z^2 + \pi^2/\Delta r^2$ where $D_* \xi_r \sim \pi \xi_r / \Delta r$, and the mode is global over a range of r where $d\Omega^2/dr < 0$ without the resonances $\bar{\omega} = 0$ or $\bar{\omega}^2 = \omega_A^2$ playing a role.

Dynamical Equations

$$\Omega = \begin{cases} \Omega_1 & r < r_1 \qquad B_a \to B\, B_a \text{ scaled} \\ \Omega_1 \left(\dfrac{r_1}{r}\right)^q & r \geq r_1 \qquad \omega_A^2 \equiv \dfrac{B^2}{4\pi\rho} \end{cases}$$

$$(k_1, k_2) = (m, k_z)$$

$$\dot{u}_1 = \frac{du_r}{dt} = -ik_1\Omega u_r + 2\Omega u_\theta + ik_2\omega_A^2 B_r$$

$$\dot{u}_2 = \frac{du_\theta}{dt} = -ik_1\Omega u_\theta - (2\Omega + r\Omega')u_r + ik_2\omega_A^2 B_\theta$$

$$\dot{u}_3 = \frac{dB_r}{dt} = -ik_1\Omega B_r + ik_2 u_r$$

$$\dot{u}_4 = \frac{dB_\theta}{dt} = -ik_1\Omega B_\theta + ik_2 u_\theta + r\Omega' B_r$$

$$\dot{u}_5 = \frac{du_z}{dt} = -ik_1\Omega u_z + ik_2\omega_A^2 B_z$$

$$\dot{u}_6 = \frac{dB_z}{dt} = -ik_1\Omega B_z + ik_2 u_z$$

Eigenvalue Problem

$$\frac{du}{dt} = Lu + N(u,u)$$

$$L = \begin{bmatrix} -ik_1\Omega & 2\Omega & ik_2\omega_A^2 & 0 & 0 & 0 \\ -2\Omega - r\Omega' & -ik_1\Omega & 0 & ik_2\omega_A^2 & 0 & 0 \\ ik_2 & 0 & -ik_1\Omega & 0 & 0 & 0 \\ 0 & ik_2 & r\Omega' & -ik_1\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & -ik_1\Omega & ik_2\omega_A^2 \\ 0 & 0 & 0 & 0 & ik_2 & -ik_1\Omega \end{bmatrix}$$

$e^{-i\omega t}$ solutions give

$$\mathrm{Det} = (\overline{\omega}^2 - k_2^2\omega_A^2)\Big[(\overline{\omega}^2 - k_2^2\omega_A^2)^2 - 2r\Omega\Omega'(\overline{\omega}^2 - k_2^2\omega_A^2) - 4\Omega^2\overline{\omega}^2\Big]$$

Local Nonlinear Model

Let the two fields be represented by

$$\mathbf{B} = B_y\,\widehat{\mathbf{y}} + \widehat{\mathbf{y}} \times \nabla\psi$$

$$\mathbf{V} = V_y\,\widehat{\mathbf{y}} + \widehat{\mathbf{y}} \times \nabla\varphi.$$

Then, we have

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} = V_y \nabla_\perp \psi - B_y \nabla_\perp \varphi + \widehat{\mathbf{y}}\widehat{\mathbf{y}} \cdot \nabla\psi \times \nabla\varphi.$$

The y–component gives

$$\frac{\partial\psi}{\partial t} + \widehat{\mathbf{y}} \times \nabla\varphi \cdot \nabla\psi = \frac{d\psi}{dt} = 0$$

for frozen flux transport from $E_\| = 0$ and the perpendicular component gives

$$\mathbf{E}_\perp = V_y \nabla_\perp \psi - B_y \nabla_\perp \varphi$$

so that

$$\frac{\partial B_y}{\partial t} = -\widehat{\mathbf{y}} \cdot \nabla_\perp V_y \times \nabla_\perp \psi + \widehat{\mathbf{y}} \cdot \nabla_\perp B_y \times \nabla_\perp \varphi.$$

Now, we can average over y and z and get

$$\frac{\partial \overline{B}_y(x)}{\partial t} = \frac{\partial}{\partial x}\overline{\left(V_y\,\frac{\partial \psi}{\partial z}\right)} - \frac{\partial}{\partial x}\overline{\left(B_y\,\frac{\partial \varphi}{\partial z}\right)}$$

as the dynamo equation for generation of $\overline{B}_y(x,t)$.

We compute the two fluxes from the linear matrix of eigenvectors

$$\overline{\widetilde{V}_y\,\widetilde{B}_x} = \overline{\left(\widetilde{V}_y\,\frac{\partial \psi}{\partial z}\right)}$$

$$\overline{\widetilde{V}_x\,\widetilde{B}_y} = \overline{\left(\widetilde{B}_y\,\frac{\partial \varphi}{\partial z}\right)},$$

from which we get α and β of the dynamo equations.

Energy Conservation

Multiply the $\partial\psi/\partial t$ by $j_y = \nabla^2\psi/4\pi$ and integrate over a volume to get

$$\frac{d}{dt}\int_{\Omega}\frac{(\nabla\psi)^2}{2}\,d^3x = \int_{\partial\Omega}\frac{\partial\nabla\psi}{\partial t}\nabla\psi\cdot d\mathbf{a} + \int_{\Omega}\nabla^2\psi[\varphi,\psi]d^3x.$$

and

$$\frac{d}{dt}\int_\Omega \frac{B_y^2}{2}\,d^3x = -\int_\Omega B_y[V_y,\psi]d^3x.$$

Finally, we compute

$$\frac{\partial V_y}{\partial t}+\mathbf{v}\cdot\nabla V_y = \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)B_y-\frac{\partial}{\partial y}\left(p+\frac{B_y^2+(\nabla_\perp\psi)^2}{8\pi}\right)$$

and the vorticity equation

$$\frac{\partial}{\partial t}\,\nabla^2\varphi+\mathbf{v}\cdot\nabla(\nabla^2\varphi) = \frac{B_y}{4\pi}\,\frac{\partial}{\partial y}\,\nabla^2\psi+[\psi,\nabla^2\psi]/4\pi$$

to compute the dynamics of φ, ψ, V_y, B_y. The linearized equations give the dispersion relation. Thus,

$$\frac{d}{dt}\int\left(\frac{B_y^2}{8\pi}+\frac{V_y^2}{2}\right)d^3x=\int\Big[-B_y[V_y,\psi]+[\psi,B_y]V_y\Big]/4\pi=0$$

and

$$\frac{d}{dt}\int\Big[\frac{(\nabla\psi)^2}{8\pi}+\frac{(\nabla\psi)^2}{2}\Big]\,d^3x = \int\Big[-\varphi[\psi,\nabla^2\psi]$$

$$+\nabla^2\psi[\varphi,\phi]\Big]/(4\pi)=0.$$

Parker Instability

$$\frac{\partial u_r}{\partial t} = (-ik_1\Omega - k^2\nu)u_r + 2\Omega u_\theta - \frac{\partial}{\partial r}\,\frac{\delta p}{\rho_0}$$

$$\frac{\partial u_\theta}{\partial t} = (-ik_1\Omega - k^2\nu)u_\theta - (2\Omega + r\Omega')u_r - \frac{1}{r}\,\frac{\partial}{\partial \theta}\,\frac{\delta p}{\rho_0}$$

$$\frac{\partial u_z}{\partial t} = (-ik_1\Omega - k^2\nu)u_z - \frac{\delta\rho}{\rho}\,g_z - \frac{\partial}{\partial z}\,\frac{\delta p}{\rho_0}$$

$$\frac{\partial}{\partial t}\,\delta p - p u_z\,\frac{\partial}{\partial z}\,\ell n\,\left(\frac{p^{1/\gamma}}{\rho}\right) = 0$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}(r u_r) + i\,k_z u_z + \frac{u_z}{\rho_0}\,\frac{\delta\rho_0}{\partial z} = 0$$

Local Dispersion Relation $\gamma = -i\overline{\omega}$

$$\overline{\omega}^2 - N^2 = \frac{k_z^2}{k_r^2}(\Omega^2 - \overline{\omega}^2).$$

Brant–Väisälä Buoyancy Frequency

$$N^2 = \frac{-g_z}{L_s} = \frac{-g_z}{L_p}\,\frac{d\ell n\, p}{d\ell n\, T}\,(\nabla - \nabla_{ad})$$

$$\nabla = \frac{d\ell n\, T}{d\ell n\, p} \quad \text{and} \quad \nabla_{ad} = 1 - \frac{1}{\gamma}$$

In atmospheric science (Gill, pp. 52–53) ∇_{ad} is often called κ. For diatomic gases (O_2, N_2) the $\gamma = 7/5$ and $\kappa = 2/7$. Potential temperature θ is defined as the temperature T a parcel of air would have if moved adiabatically to a reference pressure surface Pr. In atmospheric science $Pr = 100$ millibars $= 1$ bar is the usual reference pressure surface. The entropy per unit mass then $\eta = C_p\, \ell n\, \theta$ and $N^2 = g\, d\, \ell n\, \theta / dz$.

Toroidal and Vertical Magnetic Field

$$\frac{\partial u_r}{\partial t} = 2\Omega\, u_\theta + \frac{\mu}{4\pi\rho}\left(ik_2 B_z b_r - \frac{2B_\theta}{r}\, b_\theta\right) - \frac{\partial}{\partial r}\,\frac{\delta p}{\rho}$$

$$\frac{\partial u_\theta}{\partial t} = -(2\Omega + r\Omega')u_r + \frac{\mu}{4\pi\rho}\Big(ik_2 B_z b_\theta + (D_* B_\theta)b_r\Big)$$

$$\frac{\partial b_r}{\partial t} = ik_2 B_z u_r$$

$$\frac{\partial b_\theta}{\partial t} = ik_2 B_z u_\theta + r\Omega' b_r - r\,\frac{d}{dr}\left(\frac{B_\theta}{r}\right) u_r$$

$$\frac{\partial u_z}{\partial t} = \frac{\mu i k_2 B_z b_z}{4\pi\rho} - ik_2\,\frac{\delta p}{\rho}$$

$$\frac{\partial b_z}{\partial t} = ik_z B_z u_z$$

Nonlinear Transfer Terms

$$\frac{\partial u_r}{\partial t} = (Lu)_r - \mathbf{u} \cdot \nabla u_r + j_\theta b_z - j_z b_\theta + \frac{u_\theta^2}{r}$$

$$\equiv (Lu)_r + N_r(u, b)$$

$$\frac{\partial u_\theta}{\partial t} = (Lu)_\theta - \mathbf{u} \cdot \nabla u_\theta - \frac{u_r u_\theta}{r} + j_r b_z - j_z b_r$$

$$\equiv (Lu)_\theta + N_\theta(u, b)$$

$$\frac{\partial u_z}{\partial t} = (Lu)_z - (\mathbf{u} \cdot \nabla) u_z + j_r b_\theta - j_\theta b_r = (Lu)_z + N_z$$

$$\frac{\partial \mathbf{b}_\theta}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{u}$$

$$\mu_0 j_r = \frac{\partial b_\theta}{\partial z} \qquad \mu_0 j_\theta = \frac{\partial b_r}{\partial z} - \frac{\partial b_z}{\partial r} \qquad \mu_0 j_z = \frac{1}{r}\,\frac{\partial}{\partial r}(r b_\theta)$$

Nonlinear Transfer Fluxes

$$\Pi = \langle u_r u_\theta \rangle = u_\theta^* u_r + u_\theta u_r^*$$

$$T_M = \langle b_r b_\theta \rangle = b_r^* b_\theta + b_r b_\theta^*$$

$$T_1 = ik_z \left(u_r^* b_r - u_r b_r^* \right)$$

$$T_2 = ik_z \left(u_\theta^* b_\theta - u_\theta b_\theta^* \right)$$

$$T_3 = ik_z \left(u_z^* b_z - u_z b_z^* \right)$$

$$\frac{d}{dt}\left(\frac{u_r^2}{2}\right) = 2\Omega\Pi + \omega_A^2 T_1 - u_r^* \frac{\partial}{\partial r}\left(\frac{\delta p}{\rho}\right) + \nu u_r^* \left(\nabla^2 u_r - \frac{u_r}{r^2}\right)$$

$$\frac{d}{dt}\left(\frac{u_\theta^2}{2}\right)! = -(2\Omega + r\Omega')\Pi + \omega_A^2 T_2 + \nu u_\theta^* \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right)$$

$$\frac{d}{dt}\left(\frac{u_z^2}{2}\right) = \omega_A^2 T_3 - u_z^* \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho}\right) + \nu u_z^* \nabla^2 u_z$$

$$\frac{d}{dt}\left(\frac{b_r^2}{2}\right) = -T_1 + \eta b_r^* \left(\nabla^2 b_r - \frac{b_r}{r^2}\right)$$

$$\frac{d}{dt}\left(\frac{b_\theta^2}{2}\right) = -T_2 + r\Omega' T_M + \eta b_\theta^* \left(\nabla^2 b_\theta - \frac{b_\theta}{r^2}\right)$$

$$\frac{d}{dt}\left(\frac{b_z^2}{2}\right) = -T_3 + \eta b_2^* \nabla^2 b_z$$

$$\frac{d}{dt}\left(\frac{1}{2}\,u^2\right) = -r\Omega'\Pi - \nu\left(\mathbf{u}^* \cdot D\mathbf{u} + u \cdot D(t)\right) + \omega_A^2\,T$$

$$\frac{d}{dt}\left(\frac{1}{2}\,b^2\right) = r\Omega' T_M - \eta\left(b^* \cdot D \cdot b + b \cdot D \cdot b^*\right) - T.$$

Conservation of Energy

$$\frac{d}{dt}\left\langle\left(\frac{1}{2}\,u^2 + \frac{1}{2}\,\omega_A^2 b^2\right)\right\rangle = -r\Omega'(\Pi - T_M) - \nu(u, Du) - \eta(b, Db)$$

Steady State Turbulence

$$-r\Omega'(\Pi - T_M) = \nu(u, Du) + \eta(b, Db).$$

Turbulence may be pulsating on intermediate time scale.
From the quasilinear approximation

$$P_{in} = \frac{\widehat{\omega}_A q (2q - \widehat{\omega}_A^2)^{1/2}}{(4 - 2q + \widehat{\omega}_A^2)(4 - 2q + 2\widehat{\omega}_A^2)^{1/2}}\left\langle u_r^2 \right\rangle$$

$$q = -d\ell n\,\Omega / d\ell n\, r \qquad \text{and} \qquad \widehat{\omega}_A = \left\langle k_z^2 \right\rangle^{1/2} v_A/\Omega.$$

Eigenvalue Problem with Toroidal B_θ

Define

$$\omega_z^2 = \frac{k_z^2\,B_z^2}{4\pi\rho} \qquad \omega_\theta^2 = \frac{B_\theta^2}{4\pi\rho r^2}$$

$$\left(\overline{\omega}^2 - \omega_{Az}^2 - r\,\frac{d}{dr}\,\Omega^2 + r\,\frac{d}{dr}\,\omega_\theta^2\right)\xi_r$$

$$+2i(\overline{\omega}\Omega - \omega_\theta\omega_z)\xi_\theta = \frac{\partial}{\partial r}\,\frac{\delta p}{r}$$

$$\left(\overline{\omega}^2 - \omega_z^2\right)\xi_\theta - 2i(\overline{\omega}\Omega - \omega_\theta\omega_z)\xi_r = 0$$

$$\left(\overline{\omega}^2 - \omega_z^2\right)\xi_z = ik_z\,\frac{\delta p}{\rho}$$

Incompressibility yields

$$\left(\overline{\omega}^2 - \omega_z^2\right)D_*\xi_r = k^2\,\frac{\delta p}{\rho}$$

and the eigenvaluation equation becomes

$$\left(\overline{\omega}^2 - \omega_z^2\right)(DD_* - b^2)\xi_r = -k^2$$

$$\times\ \left[r\,\frac{d}{dr}(\Omega^2 - \omega_\theta^2) + \frac{4(\overline{\omega}\Omega - \omega_z\omega_\theta)}{\overline{\omega}^2 - \omega_z^2}\right]\xi_r.$$

For $\Omega = 0$ the system is stable if $B_\theta(r) = B_{\theta 1}(r_1/r)^{p_T}$ with $p_T > 1$ (Chandrasekhar, p. 398).

The high frequency oscillations have

$$u_r = A_2 \cos(\omega_2 t) u_r(x) \cos(k_z z)$$

$$u_\theta = A_2 \left(\frac{\delta u_\theta}{\delta u_x}\right) \cos(\omega_2 t) u_r(x) \sin(k_z z)$$

with

$$\frac{\delta u_\theta}{\delta u_x} \cong \pm \frac{2(2-q)}{\sqrt{2(2-q)+\omega_A^2}}.$$

The growing modes have

$$u_r = A_1\, e^{\gamma_1 t} u_r(x) \cos(k_z z)$$

$$u_\theta = A_1\, e^{\gamma_1 t} \left(\frac{\delta u_\theta}{\delta u_x}\right) u_r(x) \cos(k_z z)$$

$$\frac{\delta u_\theta}{\delta u_x} = \frac{1}{2}\,\gamma_1 \left(\frac{4+\omega_A^2}{4-2q+2\omega_A^2}\right) = \frac{1}{2}\,\frac{\omega_A(2q-\omega_A^2)^{1/2}(4+\omega_A^2)}{(4-2q+2\omega_A^2)^{3/2}}$$

$$\Pi = \langle u_r u_\theta \rangle = \frac{1}{2}\,\frac{\gamma_1}{\Omega} \left(\frac{4+\omega_A^2}{4-2q+2\omega_A^2}\right) \langle u_r^2 \rangle.$$

Nonlinear Transfer Terms

$$\frac{\partial u_r}{\partial t} = (Lu)_r - \mathbf{u} \cdot \nabla u_r + \frac{u_\theta^2}{r} - \frac{\Omega_1^2 r_1^3}{r^2} + j_\theta b_z - j_z b_\theta$$

$$\frac{\partial u_\theta}{\partial t} = (Lu)_\theta - \mathbf{u} \cdot \nabla u_r + \frac{u_r u_\theta}{r} - j_r b_z + j_z b_r$$

$$\frac{\partial u_z}{\partial t} = (Lu)_z - \mathbf{u} \cdot \nabla u_z + j_r b_\theta + j_\theta b_r$$

$$\frac{\partial b_r}{\partial t} = (Lb)_r - \frac{ik_2}{r}\,\varepsilon_z + ik_3 \varepsilon_\theta$$

$$\frac{\partial b_\theta}{\partial t} = (Lb)_\theta + \frac{\partial}{\partial r}\,\varepsilon_z - ik_3 \varepsilon_r$$

$$\frac{\partial b_z}{\partial t} = (Lb)_z - \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\varepsilon_\theta) + \frac{ik_2}{r}\varepsilon_r \qquad \text{with}$$

$$\mu_0 j_r = +ik_z b_z / r - ik_3 b_\theta \qquad \varepsilon_r = -u_\theta B_z + u_z B_\theta$$

$$\mu_0 j_\theta = ik_3 b_r - \frac{\partial}{\partial r}\,b_z \qquad \varepsilon_\theta = -u_z B_r + u_r B_z$$

$$\mu_0 j_z = \frac{1}{r}\,\frac{\partial}{\partial r}(r\,b_\theta) - \frac{ik_2}{r}\,b_r \qquad \varepsilon_z = -u_r B_\theta + u_\theta B_r$$

Amplitude Limits of $k_y = 0$ Modes

Nonlinear Convection Limit

$$\gamma \delta v_y = \delta v_x \, \frac{d}{dx} \, \delta v_y \simeq \delta v_z \, \frac{d}{dz} \, \delta v_y. \qquad \text{Thus}$$

$$\delta v_x = \frac{\gamma}{k_x} \sim \frac{k_z}{k_x} \left| \frac{r\Omega'}{\Omega} \right| v_A$$

$$\delta v_z = \frac{\gamma}{k_z} = \left| \frac{r\Omega'}{\Omega} \right| v_A$$

Nonlinear B-field

$$\gamma \delta B_x = \left[(B + \delta B_z) \frac{d}{dz} + \delta B_x \, \frac{d}{dx} \right] \delta v_x$$

$$\gamma \delta B_y = (B \, k_z + \mathbf{k} \cdot \delta \mathbf{B}) \delta v_y$$

$$\gamma \delta B_z = (B \, k_z + k_r \delta B_x + k_z \delta B_z) \delta v_z$$

$$\delta B_x \leq \frac{k_z}{k_x} \, B$$

$$\delta B_z = \frac{k_x}{k_z} \, \delta B_x \leq B$$

$$\delta B_y = \frac{v_y'}{\gamma} \, \delta B_x = \frac{k_z v_y'}{k_x \gamma} \, B$$

Shearing Limit for k_y-Modes

High $k_y = m/r$ modes never reach nonlinear limit due to phase mixing.

From viscosity $\nu = \mu/\rho$ or resistivity η we have the matrix system

$$\frac{du}{dt} = -i\, k_y\, r\Omega' x\, u + Lu + \eta\, \frac{\partial^2}{\partial x^2}\, u.$$

Approximate $Lu = \gamma_1 u$ where γ_1 is the linear growth rate. Transform to the local frame by $u = (\exp -i\, k_y \Omega' x t)\widetilde{u}(t)$. Then

$$\frac{d\widetilde{u}}{dt} = \left(\gamma_1 - \eta\, k_y^2 \Omega'^2 x^2 t^2\right) \widetilde{u}.$$

Take the minimum mode width $x^2 = w^2$, then growth stops at

$$t_* = \left(\gamma_1/\eta\, k_y^2 w^2 \Omega'^2\right)^{1/2}$$

and the maximum amplitude is

$$u_{\max} = u_0 \exp\left(\frac{2}{3}\,\gamma_1 t_*\right) \equiv u_0 \exp(S_m) \qquad \text{with}$$

$$S_m = \frac{2}{3}\,\gamma_1^{3/2}/(\eta\, k_y^2 w^2 \Omega'^2)^{1/2}.$$

Turbulent Reaction on Mean Flows

Exact Balance Equation

$$\frac{\partial}{\partial t}(ru_\theta) + \frac{1}{r}\frac{\partial}{\partial r}(r^2 u_r u_\theta) + \frac{\partial}{\partial \theta}(u_\theta^2) + \frac{\partial}{\partial z}(ru_\theta u_z) = \frac{r\,F_\theta}{\rho}.$$

Mean Field Equation for Angular Momentum

$$\frac{\partial}{\partial t}\overline{ru_\theta} + \frac{1}{r}\frac{\partial}{\partial r}\, r^2\overline{(\delta u_r \delta u_\theta)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\,\frac{\overline{\delta B_r \delta B_\theta}}{4\pi\rho}\right)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\langle u_r\rangle\langle u_\theta\rangle\right).$$

Steady State Transport

$$\frac{\mu_A r^2 \Delta\Omega}{\Delta r^2} = -\underbrace{\frac{\Omega}{r}\frac{\partial}{\partial r}(r^2\langle u_r\rangle)}_{\text{source from inward flow}}$$

transport of $L_\theta$from turbulence

Advantages of Spectral Codes

- Fast calculation (N log N) of nonlinear convolution terms
- Accurate representation of small space scales
- Diagonal, linear operators
- Lends itself to parallel machine architecture
- F90 code with distribution of uniform $x_i = r_i^2/R^2 = i\Delta x$ grid over processors and 2D FFT in each processor for k_y, k_z turbulent modes.

Couette-Taylor Flow Turbulence

Steady State Turbulent transport of specific angular momentum $\mathcal{L} = \Omega a^2$ from inner cylinder $r = a\eta = (2/3)b$.

Input torque T (dimensionless)

$$G = \frac{T}{\rho \nu^2 L}.$$

Kolmogorov dissipation rate ε

$$\varepsilon = \frac{(\Delta u)^3}{\ell}.$$

At large scale $\Delta u = \Omega(b - a)$ and $\ell = b - a$.

At small scale $\varepsilon = \nu k_d^2$.

Yields prediction for Torque versus Ω

$$G = \frac{\pi\eta(1+\eta)}{(1-\eta)^2}\, R_{en}^2$$

$$R_{en} = \Omega a(b - a)/\nu.$$

Lathrop, Fineberg and Swinney (1992) show

$$G \propto R^{\alpha} \quad \text{with} \quad \alpha \leq 1.87.$$

Tokamak Transport Equations

Particle Balance Equation

$$\frac{\partial\langle n_a\rangle}{\partial t}+\frac{1}{V'}\,\frac{\partial}{\partial\Psi}(V'\Gamma_a)=\left\langle\int d^3v\mathcal{I}_a\right\rangle \tag{15}$$

Energy Balance Equation

$$\frac{\partial}{\partial t}\left\langle\frac{3}{2}\,p_a\right\rangle+\frac{1}{V'}\,\frac{\partial}{\partial\Psi}\left[V'\left(q_a+\frac{5}{2}\,T_a\Gamma_a\right)\right]$$

$$=-\Pi_a\,\frac{\partial V^\zeta}{\partial\Psi}-e_a\Gamma_a\,\frac{\partial\langle\Phi_1\rangle}{\partial\Psi}+\left\langle n_a e_a\,\mathbf{u}_{a1}\cdot\mathbf{E}^{(A)}\right\rangle$$

$$-\left\langle e_a\widetilde{\Phi}_1\,\frac{\partial n_a}{\partial t}\right\rangle+\frac{1}{2}\,(V^\zeta)^2\frac{\partial}{\partial t}\langle m_a n_a R^2\rangle$$

$$+\left\langle\int d^3v\;\varepsilon\;(C_a+\mathcal{D}_a+\mathcal{I}_a)\right\rangle \tag{16}$$

Toroidal Momentum Balance Equation

$$\frac{\partial}{\partial t}\left\langle\left(\sum_a m_a n_a\right)R^2V^\zeta\right\rangle+\frac{1}{V'}\,\frac{\partial}{\partial\Psi}\left(V'\sum_a\Pi_a\right)$$

$$=\sum_a\left\langle\int d^3v\;m_a v_\zeta(\mathcal{D}_a+\mathcal{I}_a)\right\rangle \tag{17}$$

Growth Ratio Between Parker and B-H Instability

Parker growth γ_P

$$\gamma_p^2 \leq g\, k_P$$

where $g = c_s^2(1 + 1/\beta)/\gamma H$ and $k_P = (1 + \beta^{-1})^{1/2}/H$.

Balbus-Hawley growth rate

$$\gamma_B^2 \leq \frac{r\Omega'}{\Omega}\, \omega_A^2$$

where $H^{-1} \ll k_{\mathrm{BH}} \lesssim \Omega/v_A$.

Ratio

$$R = \frac{\gamma_P^2}{\gamma_{\mathrm{BH}}^2} \cong \frac{g\, k_p}{\Omega^2} \times 10.$$

Protoplanetary Disk Plasma

- Gravitational Force from Central Mass M

$$g = \frac{\mathrm{GMZ}}{(r^2+z^2)^{3/2}} \cong \frac{\mathrm{GMZ}}{r^3}$$

- Keplerian Shear $q = -r\Omega'/q = 3/2$

Gives ratio profile

$$\frac{\gamma_P^2}{\gamma_{\mathrm{BH}}^2} \leq k_P\, z \leq k_p\, H \sim \left(1+\frac{1}{\beta}\right)^{1/2}.$$

For $M = M_\odot$, $T \sim 100\,\mathrm{K}$ and $\beta \gg 1$

$$R \sim \begin{cases} 1 & \text{when } r \sim 100 \text{ AU} \\ 10^3 & \text{when } r \sim 1 \text{ AU} \end{cases}$$