WINDMI-RC: A Family of Physics Network Models for Storms

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Abstract. An important problem in magnetospheric physics is to develop integrated dynamical systems capable of modeling global storm and substorm databases for with long term aim of developing space weather forecasting tools. WINDMI-RC is a family of physics-based models that range in dimensionality d from low-order models which model the flow of energy between the dominant global energy components to high-order models with d ~ 10^2-10^3, for models that resolve the nonlinear dynamics of the system into different latitudes. The models are intrinsically three-dimensional in configuration space, and use the basic geometry of the Tsyganenko magnetic field model to define the geometrical quantities. Optimal values of the parameter vector $P = \{L, C, \Sigma\ldots\}$ str gound using the genetic algorithm on given databases. The models satisfy the constraints of the conservations laws of energy and electrical charge in their network of nodes and branches that follow the magnetic field structure of the system. The new RC-model describes the injection of plasma from the plasma sheet across the Alfvèn layer into the ring current.

1. Introduction

WINDMI model is a physics-based low-dimensional model for the coupled Solar Wind-Magnetosphere-Ionosphere system that couples the four basic energy components of the nightside magnetosphere (i) lobe magnetic energy, (ii) plasma thermal energy, (iii) parallel kinetic energy, and (iv) cross-tail perpendicular kinetic energy (Horton and Doxas 1996, 1998) to the ionosphere via the night side region-1 currents.

1.1 Multi-Mode Dynamical Model for Synthesis of Substorm Physics

The dynamics of the spatially-resolved WINDMI system is given by the discretized current-voltage-pressure vector equations:

$$L_{ij}\frac{dI_j}{dt} = V_i^{\mathrm{sw}} - V_i(t) + M_{ij}\frac{dI_j^1}{dt}\,, \qquad (1.1)$$

$$C_i\frac{dV_i}{dt} = I_i - I_i^1 - \frac{(B_i - B_{i-1})\Delta z}{\mu_0} I^{\mathrm{MHD}}(p_i) - \Sigma_i V_i\,, \qquad (1.2)$$

$$\frac{\partial B_i}{\partial t} = -\frac{V_{i+1} - V_i}{\Delta x_i L_y}\,, \qquad (1.3)$$

$$\frac{3}{2}\frac{dp_i}{dt} = \frac{\Sigma_i}{\Omega_i} V_i^2 - u_{0i}\left(\frac{K_{\|i}}{p_i\Omega_i}\right)^{1/2} \Theta(I_i - I_{i,ci}) p_i \,, \tag{1.4}$$

$$\frac{dK_{\|i}}{dt} = I^{\text{MHD}}(p_i) V_i - \frac{dK_{\|i}}{\tau_{\|}} \,, \tag{1.5}$$

$$L_{ij}^1 \frac{dI_j^1}{dt} = V - V_i^1 + M_{ij}\frac{dI_j}{dt} \,, \tag{1.6}$$

$$C_i^1 \frac{dV_i^1}{dt} = V_I - V_i^1 - \Sigma_i^1 V_i^1 \,. \tag{1.7}$$

It is straightforward to show that the rate of change of the total energy in W associated with Eqs. (1.1)-(1.7) is determined by

$$\frac{dW}{dt} = I_i V_i^{\text{sw}} - \Sigma_i^1 (V_i^1)^2 - u_{0i}\left(\frac{K_{\|i}}{\rho_i\Omega_i}\right)^{1/2} \Theta(I_i - I_{i,ci}) p_i - \frac{K_{\|}}{\tau_{\|}} \tag{1.8}$$

$$W = \frac{1}{2}\sum_{i,j}\left[L_{ij} I_i I_j - M_{ij} I_i I_j^1 + L_{ij}^1 I_j^1 \right] \tag{1.9}$$

$$+\frac{1}{2}\sum_i \left[C_i V_i^2 + C_i^1 (V_i^1)^2 \right] + \sum_i \left(\frac{3}{2} p_i \Omega_i + K_{\|i} \right).$$

In the absence of solar wind driving $(V_i^{\text{sw}} = 0)$ and ionospheric loss terms, the energy is constant of the complex dynamical system. The energy transfer coupling elements cancel in terms of six pairs for each cell. In the conservation limit the system is Hamiltonian in structure.

There are six energy components of the WINDMI model. The largest reservoir of energy is the total Magnetic Energy in the system which is given by the inductance matrices and current loops with I = cross-tail currents that are perpendicular to **B** in the geotail and I^1 = field-aligned currents into the ionosphere. Equation (1.9) states that the total system energy E is the sum of magnetic + **E** × **B** flow + plasma thermal + parallel flow energy components. The magnetic energy equals the geotail plus the MI coupling. The **E** × **B** flow energy equals the geotail convection flow plus the MI coupling driven flows.

In the Hamiltonian limit with no driving and dissipation there is a wide spectrum of eigenmodes to the system. In the lowest frequency modes the currents oscillate synchronously so that the two global WINDMI modes of approximately one hour and of 15 min. are recovered. In Fig. 1 we show the spectrum of eigenvalues and the first two eigenfunctions of a model with ???

1.1.1 Dynamics of Compressional Dipolarization Pulses

The low-dimensional WINDMI model describes the energy storage and coupling in large cells of the solar wind-magnetosphere-ionosphere system. In the SR model we break up the large geotail system with total current $I(t)$ of the global WINDMI model into subcells Ω_i that cover the lobe region from the near-Earth to the distant tail. We use the Tsyganenko model to find partitions $\{x_i\}^N$ along the geotail that contain comparable amounts of current. For example, in Fig. 1 we show the full $I \cong 20$ MA geotail current could be represented by $N = 20$ cells carrying approximately 1 MA each.

In the spatially-resolved WINDMI model, each geotail cell $\Delta x_i = x_i - x_{i-1}$ contains dynamical equations for the current $I_i(t)$, cross-tail voltage $V_i(t)$ and the component of the normal ($\hat{z}$) magnetic field $B_i(t)$. The net cross-tail current $I_i - I_i^1$ is equivalent to that cell's contribution of the lobe magnetic field $B_{xi} = \mu_0 I_i(t)/\Delta x_i$. The total northward magnetic field is

$$B_z(x_i,t) = B_{dp}(x_i) + B_i(x_i,t) \tag{1.10}$$

where B_{dp} is from the Earth's magnetic dipole. There is a convection electric field given by $E_y(x_i,t) = V_i(t)/L_y$ where L_y could be taken as i-dependent, if required. Thus, there is an Earthward-directed Poynting flux of

$$S_i = \frac{1}{\mu_0} E_y(x_i,t) B_z(x_i,t). \tag{1.11}$$

1.2 Dipolarization Alfvènic Pulse Propagation

The simulations with Eqs. (1.1)-(1.7) show pulses propagating both Earthward and tailward with speeds slower than the local Alfvèn speed. The pulses are complex structures and can have a nonlinear steepening when the capacitance in Eq. (   ) uses the local normal magnetic field in Eq. (   ). The full description of the pulses requires a discussion of the eigenmodes of the vector-matrix equations, Eqs. (1.1)-(1.7). Here we motivate and simplify the discussion of the eigenvalue problem by using a local continuum approximation in Eqs. (1.2) and (1.3).

Taking the time derivative of Eq. (1.2) to form $\partial_t (C_i \partial V_i/\partial t)$ and eliminate $\partial_t B_t$ with Eq. (1.3) and then using $(V_{i+1} - V_i)/\Delta x_i$ and $(B_{i+1} - B_i)/\Delta x \rightarrow \partial x_i$ we get the local approximate equation valid for $k_x \Delta x \ll 1$. The lower-order eigenmodes of the full system (1.1)-(1.7) follow those from this continuum limit when end boundaries are not critical and $\Delta x = L_x / N$ is small. The nonlinear, nonlocal pulse equation is

$$\frac{\partial}{\partial t}\left(C \frac{\partial V}{\partial t} \right) = \mathscr{L}^{-1} \begin{pmatrix} V - V^{\text{sw}} \\ V_1 - V \end{pmatrix} + \frac{\Delta z}{\mu_0 L_y} - \Sigma \frac{\partial V}{\partial t} - \frac{dI^{\text{MHD}}}{dp} \left(\frac{2\Sigma V^2}{3\Omega_{\text{cps}}} - u_0 \Theta P \right) \tag{1.12}$$

where $\mathscr{L}$ is the 2×2 symmetric block matrix composed of L, M, and L_1. The local waves $\delta V(t)\cos(k_x x)$ are of frequency $\omega(k_x)$ where

$$\omega^2 = C^{-1} L^{-1} + \frac{\Delta z}{\mu_0 L_y} C^{-1} k_x^2 = \omega_0^2 = 1/cL k_x^2 \bar{V}_A^2 \tag{1.13}$$

with $\omega_0^2 = 1/cL$ -- the lowest global eigenvalue and $\bar{V}_A^2 = \Delta z / \mu_0 L_y C$ -- the composite Alfven velocity made up of both the lobe magnetic field $B_\ell = \Delta z / B_x'$, and the north-south $B_n(t)$ equatorial plane magnetic field $v_A^2 = \Delta z B_x' B_n / \pi \rho_0 \mu_0$ with $\rho_0 = \Sigma_i m_i n_i (z=0)$ on the geotail axis. Here, we note that Moore *et al.* suggest that ρ_0 may contain an equal contribution of O^+ ions from ionospheric outflows if there has been earlier substorm activity.

The important feature of the compressional waves given by Eq. (B) is that there is a cast-off frequency $\omega_0 = 1/(LC)^{1/2}$ below which all fluctuations are evanescent and thus, non-propagating. Large-scale initial disturbances from a current disruption or magnetic reconnection event will then have only the smaller space scale spectral components $|k_x| > \omega_0 / \overline{v}_A^2$ launching a propagating pulse. Near the cut-off frequency the group velocity v_g vanishes and the phase velocity v_p goes to infinity with the product constant $v_g\, v_p = \overline{v}_A^2$ as is standard for TE and TM waves in arbitrarily-shaped geometry (Jackson, 1999, 3rd ed., p. 364).

Thus, in addition to the oxygen loading used by Moore *et al.* to explain the sub-Alfvenic speeds of the depolarization pulse, the large space structures will propagate with sub-Alfvenic speeds $v_A(\omega/\omega_0 - 1)^{1/2}$ low-frequency-length axial wavelengths.

The value of the cut-off period $T_0 = 2\pi(\mathscr{L}_1 C)^{1/2}$ has not been well determined yet. Rough estimates yield $T_0 \sim 1$ hr for $L \sim 50$ H and $C = 5 \times 10$ F. However, we find using the GA algorithm to optimize the performance of the reduced $d = 8$ model given in the next section that $L \sim 100$ H, $L_I = X$ and $M = Y$. Clearly, the value of L and C are dependent on the shape and size of the magnetospheric tail cavity. The shape and size of the cavity are known to increase with strong southward IMF. The relevant estimate of T_0 requires knowledge of the solar wind conditions over the period of time T0 to 2T0 startingt somewhat prior to the interval of concern. For the October 2000 GEM storm the GA optimization gives the values in Table 1 for the parameters corresponding to $2\pi(LC)^{1/2} = 1.6\text{hr}$ and $\omega_0 = X.X$.

1.2 Global WINDMI Model

The base WINDMI model is a six-dimensional 13-parameter system, given by:

$$L\frac{dI}{dt} = V_{\text{sw}}(t) - V + M\frac{dI_1}{dt} \tag{1.14}$$

$$C\frac{dV}{dt} = I - I_1 - I_{\text{ps}} - \Sigma V \tag{1.15}$$

$$\frac{3}{2}\frac{dp}{dt} = \Sigma\frac{V^2}{\Omega_{\text{cps}}} - u_0 K_{\parallel}^{1/2}\Theta(I - I_c)p \tag{1.16}$$

$$\frac{dK_{\parallel}}{dt} = I_{\text{ps}}V - \frac{K_{\parallel}}{\tau} \tag{1.17}$$

$$L_I\frac{dI_1}{dt} = V - V_I + M\frac{dI}{dt} \tag{1.18}$$

$$C_I \frac{dV_I}{dt} = I_1 - \Sigma_I V_I \tag{1.19}$$

The current loops $I_i(t)$ now interact through the magnetic flux from the time-dependent magnetic dipole loop $A_i\, I_i$ at x_i through dipole loop $A_j\, I_j$ at x_j. This interaction is described by the geotail inductance matrix L_{ij} with

$$L_{ij} = \frac{\mu_0 A_i A_i}{\left[(x_i - x_j)^2 + R^2\right]^{3/2}} \tag{1.20}$$

and the total lobe magnetic energy in the geotail and (*gt*) is given by

$$W_M^{gt} = \frac{1}{2}\sum_{i,j}^{N} L_{ij} I_i I_j . \tag{1.21}$$

The nine physical quantities $L, C, \Sigma, \mu, u_0, \Omega_{\mathrm{cps}} L_I, C_I$ and Σ_I are the magnetospheric and ionospheric inductance, capacitance, and conductance respectively. M is the mutual inductance between the geotail flux and the FAC 1 current loop closing in the night-side auroral ionosphere. The plasma sheet pressure gradient driven current is given by $I_{\mathrm{ps}} = \alpha p^{1/2}(t)$ as derived from force balance and Ampere's law. The parameter α is an average over the pressure profile in the current sheet with an approximately value readily computed from total pressure balance. The solar wind driving voltage V_{sw} in Eq. (1.1) is the input time series for this nonlinear driven-dissipative system. The central plasma sheet unloading term $u_0 K_{\|}^{1/2} \Theta(I - I_c) p$ in Eq. (1.3) represents the rapid unloading of the stored plasma energy when the current exceeds a critical value, I_c. The trigger value I_c can be taken from the basic instability criteria of magnetic reconnection or of the current diversion instability (Liu and Yoon). The parallel heat flux limit that is neglected in the MHD closure (Horton and Doxas, 1996) determines the loss parameter u_0. In the absence of driving ($V_{\mathrm{sw}} = 0$) and damping ($\Sigma_I = 0$), and below the unloading limit $(I < I_c)$ the total energy is conserved. All parameters of the model can be calculated explicitly from their integral definitions given the plasma conditions in the tail. Examples of these calculations for the Tsyganenko models are in Horton, *et al.* (1998).

The dynamical system described by Eqs. (1.1)-(1.6) is derived from the Vlasov equation with a heat-flux limit closure scheme (Horton and Doxas, 1996). The essential non-MHD physics included in the model is given by:

1) The collisionless large ion transport for the geomagnetic tail, which appears in the conductivity, Σ, and transfers energy between Eqs. (1.2) and (1.3). The large ion gyroradius conductivity gives a finite conductance and nonadiabatic ion energization in the quasineutral sheet which vanishes in the MHD limit or the gyroradius vanishes. The conductivity was derived from theory and particle simulations (Horton and Tajima 1991a, 1991b) and contains the Lyons and Speiser (1982) energization mechanism for the transient ions.

2) The kinetic loss rate of thermal energy which is described by the parallel heat flux, represented by the heat flux limit parameter u_0, and the mean parallel flow velocity associated with the MHD parallel flow kinetic energy, $K_{||}$ [Eq. (1.3)]. The Geotail particle data, analyzed with respect to the parallel thermal flux by Hoshino *et al.* (1998), show that the minimum ratio of the thermal plasma energy density p to the kinetic energy density $(1/2)\rho v^2$ found in the central plasma sheet is consistent with a parallel heat flux $q_{||}$ taken as a fraction of $pv_{||}$, as in Eq. (1.3).

For WINDMI to be used as a forecasting tool, the parameters of the model need to span a range of values corresponding to the range of shapes and states of the magnetosphere. The central plasma sheet capacitance, *C*, for instance, is the volume integral of $\mathbf{E} \times \mathbf{B}$ kinetic energy in the current sheet for the cross-tail voltage *V(t)*. The unloading parameter, u_0, is the surface integral of the heat flux over the boundary of Ω_{cps}. Fast particle simulations are also useful tools for finding the range of the parameter values using their integral definitions (Horton and Doxas, 1996).

For each model **P** the output of the model is then compared to well known datasets (e.g. Blanchard and McPherron, 1993; Bargatze *et al.,* 1985). We consistently find that, when using this procedure, WINDMI produces good agreement with isolated substorms (Doxas *et al*., 1999; Doxas *et al.,* 2002).

Once the parameter ranges are established, the problem of finding an optimal model, or more relevant, a set of robust models, is ideally suited to the use of genetic algorithms.

Figure 1 gives the overall geometry of the model, including the ring current and partial ring current which will be addressed in the next section. Although it is tempting to consider WINDMI as a circuit model, it should be noted that Eqs. (1.1)-(1.4) are derived from the closure of the moments of the Vlasov equation (cf. Horton and Doxas, 1996, 1998) and that their equivalent circuit is due to the existence of conservation laws. A wide variety of physical, biological and engineering systems can be described by networks with the mathematical tool of algebraic topology (Lefschetz). In particular, the non-MHD closure scheme adds an energy unloading term [second term on the right-hand side of Eq. (1.3)] which is critical for modeling a class of internally- triggered events that cannot be reproduced by linear circuit models (cf. Weigel *et al.*, 2003).

Figure 1.

In Section 2.2 we add the ring current I_{rc}(t) and the key energy reservoir W_{rc}(t). In Section 3 we will make estimates for the values of some of the model parameters based on the Tsyganenko model. Finally, in Section 4 we will make a connection to existing models like the Burton model (Burton *et al*., 1975) and the Temerin and Li (2002) model.

2. Adding Ring Current Energy W_{rc} and the D_{st} Prediction to WINDMI

Consider the magnetosphere as the union $\cup$ of the volumes of the lobe $W_{\text{rc}}(t) = (3/2)\Omega_{\text{rc}} p_{\text{cps}} v_x.$, the central plasma sheet Ω_{cps}, the small transition (Alfvèn) volume

$\Omega_{\rm tr}$ surrounding the separatrix (Alfvèn layer) between the ring current region and the outer magnetospheric plasma, and the volume of the ring current itself, $\Omega_{\rm rc}$. The volume of the magnetosphere is then $\Omega_{\rm mag} = \Omega_{\rm rc} \cup \Omega_{\rm tr} \cup \Omega_{\rm cps} \cup \Omega_{\ell}$. Now, the total plasma energy is decomposed as $W_p = \Sigma_a \int_{\Omega_s} d^3x(3/2)p = (3/2)p_{\rm cps}\Omega_{\rm cps} + (3/2)p_{\rm rc}\Omega_{\rm rc}$, since the particle pressure is negligible in the lobe plasma and the transition layer volume is a small fraction of $\Omega_{\rm cps}$ and $\Omega_{\rm rc}$. The Dessler-Parker relation (Dessler and Parker, 1959; Sckopke, 1966) gives the D_{st} index through the total ring current energy component $W_{rc}(t) =$ (3/2) $\Omega_{\rm rc} p_{\rm rc}$ The principal source of $W_{\rm rc}(t)$ is the fraction of the central plasma sheet flux $p_{\rm cps} v_x$, that crosses the Alfvèn layer. Through test particle integration we define the effective areas $A_{\rm eff}$ for the scattering or admittance of CPS plasma across the Alfvèn layer. This gives a source of power $P_{\rm rc}$ for the ring current.

The Dessler-Parker relation

$$\frac{\Delta B_{\rm particles}}{B_E} = -\frac{\mu_0}{2\pi}\frac{W_{\rm rc}(t)}{B_E^2 R_E^3} \tag{2.1}$$

is a robust relation that for change $\Delta B_{\rm particles}$ is used for the model to give the ring current plasma energy ΔB_z from the $W_{\rm rc}(t)$. The ΔB_z is closely related to the magnetometers used for the D_{st} magnetic index. We note that there are other contributions to D_{st} as given by Liemohn (2003). Thus the ring current $I_{\rm rc}(t)$ and the associated induction electric field $E_{\rm rc}(t)$ from the ring current inductance $L_{\rm rc}$ are conjugate variables for the inner magnetosphere electrodynamics. In Eq. (2.1), $B_E = 3.1 \cdot 10^{-5}\,\mathrm{T}$ is the magnetic field strength of the Earth's internal dipole field at the Earth's surface on the equator.

Both protons and electrons are organized as they drift Earthward in the magnetotail during the storms. The Lorentz force gives the exact orbits of the particle motions, $\mathbf{F}_L = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Averaging over the cyclotron gyrations gives the guiding center orbits sufficient for the electron energization. The duskward electric field $E_y = V(t)/L_y$ is time and space varying due to the time variation of the solar wind and the projection of the E_y vector on the plane perpendicular to the local magnetic field. This projection is important to give zero parallel electric field. The physics of this process of $E_{\parallel}$ is that to the first order, electrons move so as to cancel the parallel component of the dynamo electric field.

The fractional diversion of the particles across the separatrix formed by the corotation electrostatic field and the storm time varying convection electric field can be considered as a scattering and trapping problem for ensembles of electron and ion kappa distributions. Such a process can then be modeled by an effective admission cross-section $A_{\rm eff}$ for the fraction of the particles that cross the separatrix. Of course this fraction depends on the nature of the storm time electric field and the state of the transitional region of the transitional magnetic field so only a rough characterization of the trapping fraction is possible from theory. The parameter $A_{\rm eff}$ is then one of the critical parameters of the model, and can be either estimated from theory (e.g. by Monte Carlo simulations) or viewed as a parameter to be optimized. A schematic of the ring current and its position relative to the other currents in the model is shown in Figure 1. The effective aperture, A_{eff}, is shown schematically as the cross-section that defines the fraction of the plasma sheet particles that cross in to the ring current plasma.

We integrate the Poynting flux over the toroidal volume Ω_{rc} with the aperture A_{eff} in the midnight sector of its surface $\partial\Omega_{\mathrm{rc}}$ to obtain

$$\frac{\partial}{\partial t}\int_{\Omega_{\mathrm{rc}}}\left(\frac{1}{2}\rho v_E^2+\frac{B^2}{2\mu_0}+p_\perp\right)d^3x=\int_{A_{\mathrm{eff}}} p v_{xE}\cdot da-\int_{\Omega_{\mathrm{rc}}}\mathbf{j}_{\mathrm{gc}}\cdot\mathbf{E}d^3x, \tag{2.2}$$

where j_{gc} is the guiding drift current density, and v_{E} the $\mathbf{E}\times\mathbf{B}$ drift velocity. To derive the energy density in Eq. (2.2) that contains the $\mathbf{E}\times\mathbf{B}$ flow and the plasma pressure, the microscopic form of the Poynting theorem is transformed with $\mathbf{j}=(\rho/B^2)(d\mathbf{E}_\perp/dt)+\nabla\times\mathbf{M}+\mathbf{j}_{\mathrm{gc}}$ where ρ is the mass density and $\mathbf{M}=-p_\perp\mathbf{B}/B^2$ the magnetization. Both polarization and magnetization currents arise from the gyro-orbits. The power $\int\mathbf{j}_{\mathrm{gc}}\cdot\mathbf{E}d^3x$ is the sum of: (1) the energization of the plasma by the convection electric field outside the separatrix where $\Delta P_{\mathrm{gc}}=\int j_{\mathrm{y,gc}}E_y d^3x=I(t)V(t)$, and (2) the energization of the ring current particles

$$\frac{dW_{\mathrm{rc}}}{dt}=\int\mathbf{j}_{\mathrm{rc}}\cdot\mathbf{E}d^3x=I_{\mathrm{rc}}(t)\Delta V_{\mathrm{rc}} \tag{2.3}$$

inside the separatrix, where ΔV_{rc} is the electromotive potential across the foot points of the partial ring current.

The ring current energization dW_{rc}/dt can be rewritten in the form of the power expended by the dynamo field in moving the plasma inward up against the expansion force $\mathbf{F}=-(p_\perp/B)\nabla B-p_{\|}\boldsymbol{\kappa}$ where $\boldsymbol{\kappa}=(\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}}$ is the magnetic curvature. This work-displacement relation follows after we observe that $\mathbf{j}_{\mathrm{gc}}\cdot\mathbf{E}=\Sigma_{\mathrm{rc}}\left(\mathbf{F}\times\mathbf{B}/B^2\right)\cdot\ \mathbf{E}=-\Sigma\mathbf{v}_E\cdot\ \mathbf{F}\cong-v_xF_x$. Thus the work done by the external worker - here the central plasma sheet is done to compress the plasmas into the ring current. For an inward motion $\Delta x>0$ we integrate, with the approximation of constant $p_\perp/B$ and $p_{\|}\kappa_x$, to get $\Delta W_{\mathrm{rc}}=(p_\perp/B)(B_f-B_i)+p_{\|}(\Delta x/R_c)$ where $\Delta x>0$ for displacement toward the Earth.

Thus, the projection of the relevant partial differential equations and atomic physics onto the ring current energy balance equation gives the ordinary differential equation

$$\frac{dW_{\mathrm{rc}}}{dt}=I_{\mathrm{rc}}(t)\Delta V_{\mathrm{rc}}+\frac{p_{\mathrm{cps}}(t)V(t)A_{\mathrm{eff}}}{B_{\mathrm{tr}}L_y}-\frac{W_{\mathrm{rc}}}{\tau_{\mathrm{rc}}}, \tag{2.4}$$

where τ_{rc} is the decay time which is dominated by charge exchange (Chen and Shulz, 1996; see estimates in Section 3), and $V(t)A_{\mathrm{eff}}/L_{\mathrm{y}}B_{\mathrm{tr}}$ is the injection from the dawn-to-dusk potential drop in the central plasma sheet. A_{eff} is the area of the aperture presented by the Alfvèn layer for the entrance of CPS plasma into the inner magnetosphere for the particular storm. We discuss the modeling of this aperture in the next section. Clearly, the aperture is varying during a storm, so using an average value is a first-step model.

Including the equation for the I_2 current loop together with Eq. (2.4) into the WINDMI model we obtain:

$$L\frac{dI}{dt} = V_{\mathrm{sw}}(t) - V + M\frac{dI_1}{dt} \tag{2.5}$$

$$C\frac{dV}{dt} I - 1_1 - I_{\mathrm{ps}} - \Sigma V \tag{2.6}$$

$$\frac{3}{2}\frac{dp}{dt} = \frac{\Sigma V^2}{\Omega_{\mathrm{cps}}} - u_0 K_{\|}^{1/2}\Theta(I - I_c) - \frac{pVA_{\mathrm{eff}}}{\Omega_{\mathrm{cps}} B_{\mathrm{tr}} L_y} \tag{2.7}$$

$$\frac{dK_{\|}}{dt} = I_{\mathrm{ps}}V - \frac{K_{\|}}{\tau} \tag{2.8}$$

$$\mathrm{L}_I\frac{dI_1}{dt} = V - V_I + M\frac{dI}{dt} \tag{2.9}$$

$$\mathrm{C}_I\frac{dV_I}{dt} = I_1 - I_2 - \Sigma_I V_I \tag{2.10}$$

$$L_2\frac{dI_2}{dt} = V_I - (R_{\mathrm{prc}} + R_{\mathrm{A2}})I_2 \tag{2.11}$$

$$\frac{dW_{\mathrm{rc}}}{dt} = R_{\mathrm{prc}}I_2^2 + \frac{pVA_{\mathrm{eff}}}{B_{\mathrm{tr}}L_y} - \frac{dW_{\mathrm{rc}}}{\tau_{\mathrm{rc}}} \tag{2.12}$$

where the D_{st} signal is given by

$$\mathrm{D}_{\mathrm{st}}(t) = -\frac{\mu_0}{2\pi}\frac{W_{\mathrm{rc}}(t)}{B_E R_E^3}. \tag{2.13}$$

Here L_2 is the inductance of the region 2 current loop, and R_{prc} and R_{A2} are the resistances of the partial ring current and the region 2 foot point in the auroral region. The system of Eqs. (2.5)-(2.12) is the WINMDI-RC model, a $d = 8$ system of equations for the solar wind driven coupled Inner Magnetosphere, Ionosphere, and Tail. Together with Eq. (2.13), the model gives a prediction of the D_{st} index while conserving energy in the coupled Inner-Magnetosphere-Tail system. Here we conclude with the following estimates and observations.

From the Siscoe (1982) model of the R1-R2 auroral coupling region, we infer nominal-to-maximum currents and voltages as $I_1 \sim 1\ \mathrm{MA}$, $I_2 \sim 500\ \mathrm{kA}$, $V_I \sim 100\ \mathrm{kV}$, and $R_{\mathrm{prc}}I_2 \simeq 50\ \mathrm{kV}\text{-}100\ \mathrm{kV}$ so that the partial ring current energization power is of order 25-50 GW. Stern (1983) gives an estimate of combining Region 1 and Region 2 currents of 300 GW so that 30 GW would only be 10% of the Stern's power estimate.

The CPS injection is highly dependent on the dynamical activity in the geomagnetic tail. Nonetheless, we may estimate bounds and an average value for the aperture. For $A_{\rm eff} = R_E^2$ $= 3.6\times10^{13}\,{\rm m}^2$ with $p_{\rm cps} \sim 5$ nPa and $v_x = 100$ km/s and the flux for injection into the inner magnetosphere is $pv_x = 5\times10^{-4}\,{\rm W/m}^2$ with $P_{\rm inj} \sim 20$ GW. With $A_{\rm eff}$ growing to $10R_E^2$ there are 200 GW entering the inner magnetosphere. We discuss methods known for calculating the injection process in detail in the next section.

3. Tsyganenko Ring Current, and Tail Current

A detailed version of the Dessler-Parker relation follows from Tsyganenko's model using an analytic expression for the ring current vector potential, A_ϕ, in cylindrical coordinates, where $\rho = (X_{\rm gsm}^2 + Y_{\rm gsm}^2)^{1/2}$ is the radial distance, and ϕ is the angle about the z-axis and corresponds to the longitude. The model for $A_\phi = C\rho/(\rho^2 + z^2 + 4\rho_0^2)^{3/2}$ produces a well-localized current distribution centered at ρ_0 and decreasing as $(\rho_0/\rho)^6$ for $\rho \gg \rho_0$. Clearly, the model applies for $(\rho^2 + z^2)^{1/2} \gg R_E$ and neglects the noon-midnight asymmetry. We know Tsyganenko's parameterization of the constants C and ρ_0 as a function of $K_{\rm p}$. We determine the total ring current through $\mu_0 I_{\rm rc} = C/\rho_0^2$ and the stored energies $W_{\rm rc}$ and $W_{\rm rc}^B$, where $W_{\rm rc}^B$ is the magnetic energy stored in the ring current. Parameter values were given through a maximum value of $K_{\rm p}$ = 5 in Tsyganenko (1987); for this work we extended these values using a linear fit through a maximum value of $K_{\rm p}$ = 9. We can calculate the magnetic field according to $\mathbf{B} = {\rm curl}\, A_\phi \hat{\phi} = \nabla(\rho A_\phi)\times\nabla\phi$ and determine the $\Delta B_z(\rho = R_E, z = 0)$ for the model $D_{\rm st}$. The contribution of the geotail current to the $D_{\rm st}$ is approximately 20-25% according to Turner *et al.* (2000).

The current distribution of the ring current may be obtained from the vector potential of Eq. (2.4), by $\mathbf{J} = -1/\mu_0 \nabla^2 \mathbf{A}$ since $\nabla\cdot\mathbf{A} = 0$. The ring current density is

$$J_\phi = 60 I_{\rm rc}(t)\rho_0^4/(\rho^2 + z^2 + 4\rho_0^2)^{7/2}. \tag{3.1}$$

The maximum of the current density is located at $\rho_{\max} = \sqrt{2/3}\rho_0, z = 0,$ which occurs at about 4.2 R_E for $K_p = 0$ and decreases to $2.5R_e$ for $K_p = 9$.

From the value for B_z evaluated at $\rho_0 = 4.9R_E$ we see that the relationship between the current and the $D_{\rm st}$ index is $D_{\rm st} = \mu_0 I_{\rm rc}/(4\rho_0)$. For the reference values of $I_{\rm rc} = 4$ MA and $\rho_0 = 4.9R_E$ this gives $D_{\rm st}$ = -40 nT.

The magnetic energy, $W_{\rm rc}^B$, stored in the ring current is given by $W_{\rm rc}^B = 1/2\int \mathbf{J}\cdot d\mathbf{A}.$ Using Eq. (3.1), we obtain $W_{\rm rc}^B = (5/512)\mu_0\rho_0 I_{\rm rc}^2.$ The magnetic energy and the $D_{\rm st}$ index are therefore related by $D_{st} = (512\mu_0 / 20\pi^2 C)W_{rc}^{B}$, where $C = I_{rc}\pi\rho_0^2$. The self-inductance of the ring current is given by $L_{rc} = 2W_{rc}^{B} / I_{rc}^2$. Using $I_{rc} = C / \pi\rho_0^2$ and W_{rc}^{B} gives the inductance $L_{rc} = 5\pi^2 / 256\mu_0\rho_0 \simeq 8\text{H}$ at $\rho_{max} = 4.2R_E$. For comparison, the inductance of a tokamak is $L = \mu_0 R[\log(R/a) + 1/4]$, where *R* is the major radius and *a* is the minor radius. Applying the formula to the ring current with $R = 4\ R_E$ and $a = 2\ R_E$ gives an approximation for the inductance of the ring current of about $L_{rc} = 8$ H. The effect of the large aspect ratio approximation is that the inductance scales logarithmically with respect to the aspect ratio *R/a* of the current profile model.

For the network equations, it is important to know the self-inductance L_{rc} of the ring current. We have computed the total ring current I_{rc}, the total magnetic energy W_{rc}^{B} from I_{rc} and the self-inductance from the Tsyganenko magnetic field model as a function of the D_{st}. We see that for D_{st} = -40 nT, we have $L_{rc} = 8$ H., and $I_{rc} = 4$ MA, so that $W_{rc}^{B} = 6\times10^{12}$ J.

The summation of all the energy loss processes is thought to be dominated by charge exchange losses on the O^+ and H^+ from the atmosphere (Chen and Schulz, 1996) with $\tau_{rc} \sim 1/2$ day $= 4.32\times10^4$ s. The stored ring current energy from Eq. (2.4) at equilibrium would be $W_{rc} = (P_2 + P_A)\tau_{rc} = 25\ \text{GW}\times8.64\times10^4\text{s} = 3.5\times10^{15}$ J. Thus, $W_{rc} \gg W_{rc}^{B}$ so that the estimate of the plasma energy appears high from the point of view of laboratory MHD equilibrium and stability.

Table 1 lists the parameters of the model, with estimates of typical values. Figure 2 shows the results of the WINDMI-RC model for the storm of May 15, 1997. The values of the model parameters used in that particular simulation are given in parenthesis in Table 1. We see that the model gives good agreement with observations for reasonable values of the parameters.

4. Connection to Existing Models

Equations (2.5)-(2.12) should be compared with the Burton equation (Burton *et al.*, 1975), and a recent extension of the Burton equation due to Temerin and Li (2002). The Burton *et al.* (1975) formulation gives the D_{st} as

$$D_{st} = D_{st}^{rc} + b\sqrt{p_{dyn}} - c. \tag{4.1}$$

The D_{st} is a sum of three terms: the contribution from the ring current, the contribution from the magnetopause current, which is dependent on the dynamic pressure p_{dyn} and the quiet condition represented by *c*. The ring current contribution to the D_{st}, D_{st}^{rc} is

$$\frac{d}{dt} D_{st}^{rc} = F(E_{sw}) - aD_{st}^{rc}, \tag{4.2}$$

where $F(E_{sw})$ is the injection of energy into the ring current which depends on the solar wind electric field E_{sw} , and a is the inverse decay time of the ring current. Equations (4.2) and (4.3) can be combined to give the total change of the D_{st} which is known as the Burton equation.

The Temerin-Li formulation represents the D_{st} as a sum of several terms:

$$D_{st} = d_{st}^1 + d_{st}^2 + d_{st}^3 + \text{Pressure Term} + f\left(B_z^{\rm IMF}\right)C. \tag{4.3}$$

The *Pressure Term* is not given only by $\sqrt{p_{\rm dyn}}$ as in the Burton *et al.* (1975) formulation, but includes a dependence on the solar wind density. The three terms d_{st}^t are introduced empirically where d_{st}^1 represents the ring current contribution as done by Burton and $d_{st}^2 + d_{st}^3$ "probably represents some combination of the so-called partial ring current and the tail currents" (Temerin and Li, 2002). The key new physics in the WINDMI-RC model is the effective aperture, $A_{\rm eff}$, which will need to be parameterized in terms of $B_z^{\rm IMF}$, similar to the $f(B_z^{\rm IMF})$ term in Eq. (3.4).

5. Conclusions

A dynamical model of the coupled Inner Magnetosphere Tail has been constructed by considering the coupling of the ring current energy to the central plasma sheet, and using the Dessler-Parker relation to compute the D_{st} from the ring current energy. The effective aperture $A_{\rm eff}$ for transferring energy from the CPS through the Alfvèn separatrix layer to the ring current can be either estimated or used as a parameter to be optimized.

The major advantage of this proposed network model is that the separate components of the magnetospheric current system, such as the ring current, tail current, and partial ring current, are introduced explicitly and not in an empirical manner. It is important to determine the relative contributions of these systems to the total D_{st}. For example, Turner *et al*. (2000) determine the contribution from the tail current to be approximately one quarter of the total D_{st}.

The key advantage of the physics-derived energy and charge-conserving networks is that the flow of energy can be tracked and cataloged for different types and strengths of storms. It may well be that the network model will give a useful intermediate level tool for interpreting (ordering) storm data. Contracting a large storm observational database to a smaller physically-ordered database would be of considerable benefit to MHD modelers. In any event, we argue that the network method of describing the complex, three-dimensional magnetosphere is a mathematically and physically sound approach that needs to be developed.

FIGURE

A schematic of the current systems treated by the model. $A_{\rm eff}$ is the effective area of the aperture presented by the Alfvèn layer for the entrance of CPS plasma into the inner magnetosphere [cf. Eq. (2.4)]. I_2 is the partial ring current.

FIGURE

The D_{st} index (heavy line) and the WINDMI-RC response (dots) for the storm of May 15, 1997. The value of effective aperture is $A_{\rm eff} = 7.7\times 10^{14}/(B_{\rm tr}\, L_y)$ [cf. Eq. (2.12)], which corresponds to $A_{\rm eff} = 2.0\ R_E^2$.

Table 1. WINDMI Parameters

V^{sw}	Solar wind input voltage from interplanetary electric field $V_x^{sw} B_s$ and effective magnetopause interaction length L_y^{eff}. $V_{sw} = V_x^{sw} B_s L_y^{eff}$ in volts.
L (97.4 H)	Inductance of the lobe cavity surrounded by the geotail current *I(t)*. The nominal value is L = $\mu_0 A_\ell / L_x^{eff}$ in Henries where A_ℓ is lobe area and L_x^{eff} the effective length of the geotail solenoidal. Computation of L as function of the IMF from Tsyganenko are given in Horton *et al.* (1998).
C (4800 F)	Capacitance of the central plasma sheet in Farads. The nominal value is $C = \rho_m L_x L_z / B^2 L_y$ where ρ_m is the mass density in kg/m^3, $L_x L_z$ is the meridional area of the plasma sheet, L_y the dawn-to-dusk width of the CPS and B the magnetic field on the equatorial plane. Computations of C are given in Horton and Doxas (1996).
Σ (7.8 mho)	Large gyroradius ρ_i plasma sheet conductance from the quasineutral layer of height $(L_z \rho_i)^{1/2}$ about the equatorial sheet. The nominal value of Σ = 0.1 $(n_e/B_n)(\rho_i L_z)^{1/2}$. Computation of Σ is given in Horton and Tajima (1992).
$\Omega(10000 R_E^3)$	Volume of the central plasma sheet that supports mean pressure $p(t)$. In some works $\Omega = \Omega_{cps}$.
$u_0 (4.2\times10^{-9})$m/kg$^{1/2}$	Heat flux limit parameter for parallel thermal flux on open magnetic field lines $q_{\parallel} = u_0 (K_{\parallel} / \rho_m)^{1/2}$ where $(K_{\parallel} / \rho_m)^{1/2}$ is the mean parallel flow velocity.
τ (10 min)	Confinement time for the parallel flow kinetic energy K in the central plasma sheet.
$I(p)$	The geotail current driven by the plasma pressure p confined in the central plasma sheet. Pressure balance between the lobe and the central plasma sheet gives $B_\ell^2 / 2\mu_0 = p$ with $2L_x B_\ell = \mu_0 I_p$. This defines the coefficient α in $I_p = \alpha(p)^{1/2}$ to be approximately $\alpha = 2.8 L_x / \mu_0^{1/2}$.
L_1 (19 H)	The self-inductance of the wedge current or the nightside region 1 current loop $I_1(t)$.
M (2/3 H)	The mutual inductance between the nightside region 1 current loop I_1 and the geotail current loop I.
L_2 (8 H)	The inductance of the ring current.
C_1 (800 F)	The capacitance of the nightside region 1 plasma current loop.
Σ_I (3 mho)	The ionospheric Pedersen conductance of the westward electrojet current closing the I_1 current loop in the auroral (~ 100 km, 68°) ionosphere.
R_{prc} (0.1 ohm)	The resistance of the partial ring current.
τ_{rc} (12 hrs)	The decay time for the ring current energy.

Acknowledgements

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